Extensions of semilattices of groups arising from partial actions of groups

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[1]Partially supported by CNPq of Brazil: 307659/2009–7
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Partial Actions and Representations Symposium,
Gramado, Brazil,
May 11–15, 2014

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The category $\mathrm{pMod}(G)$ is not abelian in general, because $\mathrm{Hom}(A, A') = \emptyset$ for some $A, A' \in \mathrm{pMod}(G)$. For example, this happens when $1_x = 1_y$ in A, but $1'_x \neq 1'_y$ in A' for some $x, y \in G$.

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The map δ^n is a homomorphism $C^n(G, A) \to C^{n+1}(G, A)$, such that $\delta^{n+1}\delta^n = 0$ for all $n \geq 0$.

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Proposition

For any $n \geq 0$ the map $A \mapsto H^n(G, A)$ is a functor from $\mathrm{pMod}(G)$ *to the category* Ab *of abelian groups.*

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The above proposition shows that it is enough to study cohomology with values in inverse partial G-modules.

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(ii) $\lambda_s(\alpha(e)) = \alpha(ses^{-1})$ for all $s \in S$ and $e \in E(S)$.

Definition (Lausch, 1975)

A morphism of S-modules $\varphi : (A, \lambda, \alpha) \to (A', \lambda', \alpha')$ is a homomorphism of semigroups $A \to A'$, such that

(i) $\varphi \circ \alpha = \alpha'$ on $E(S)$;

(ii) $\varphi \circ \lambda_s = \lambda'_s \circ \varphi$ on A for all $s \in S$.

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Partial homomorphisms

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It follows that $\Gamma(x)\Gamma(x^{-1})$ is an idempotent, which will be denoted by e_x.

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Remark

If such a partial homomorphism $\Gamma : G \to S$ exists, then property (i) guarantees that S is inverse.

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The map $x \mapsto 1_x\delta_x$ is an admissible partial homomorphism $\Gamma^\theta : G \to E(A) *_\theta G = S^\theta$ associated with (A, θ).

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Partial cohomology of groups as cohomology of inverse semigroups

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An extension of A by G is a pair consisting of an admissible partial homomorphism $\Gamma : G \to S$ and an extension (in the sense of Lausch) of A by S.

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Two extensions $\Gamma : G \to S$, $A \xrightarrow{i} U \xrightarrow{j} S$ and $\Gamma' : G \to S'$, $A \xrightarrow{i'} U' \xrightarrow{j'} S'$ of A by G are called equivalent

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$$\begin{array}{ccc} G & \xrightarrow{\Gamma} & S \\ & \searrow^{\Gamma'} & \downarrow \nu \\ & & S' \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{i} & U & \xrightarrow{j} & S \\ \| & & \downarrow \mu & & \downarrow \nu \\ A & \xrightarrow{i'} & U' & \xrightarrow{j'} & S' \end{array}$$

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Let $\Gamma : G \to S$, $A \xrightarrow{i} U \xrightarrow{j} S$ be an extension of A by G and θ the corresponding partial action of G on A. Then there is an equivalent extension of the form $\Gamma^\theta : G \to S^\theta$, $A \xrightarrow{i} U \xrightarrow{j'} S^\theta$.

Extensions of (A, θ) by G

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Corollary

Equivalence classes of extensions of (A, θ) by G are in a one-to-one correspondence with elements of $H^2(G, A)$.

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Theorem

Any extension of (A, θ) by G is equivalent to $A *_{\theta, f} G$ for some (unique up to a coboundary) $f \in Z^2(G, A)$.

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An extension $A \xrightarrow{i} U \xrightarrow{j} S$ of (A, θ) by G is said to split if there is a homomorphism $k : S \to U$ (called a splitting) such that

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Proposition

An extension of (A, θ) by G splits if and only if it is equivalent to $A *_\theta G$.

The splittings of extensions of (A, θ) by G

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Lemma

The splittings of the extension $A *_\theta G$ are in a one-to-one correspondence with the elements of $Z^1(G, A)$.

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Two splittings k_1 and k_2 of a split extension $A \xrightarrow{i} U \xrightarrow{j} S$ of (A, θ) by G are said to be A-conjugate

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Two splittings k_1 and k_2 of a split extension $A \xrightarrow{i} U \xrightarrow{j} S$ of (A, θ) by G are said to be *A-conjugate* if there is $a \in A$ such that $k_1(s) = i(a)k_2(s)i(a)^{-1}$ for all $s \in S$.

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Two splittings k_1 and k_2 of a split extension $A \xrightarrow{i} U \xrightarrow{j} S$ of (A, θ) by G are said to be *A*-conjugate if there is $a \in A$ such that $k_1(s) = i(a)k_2(s)i(a)^{-1}$ for all $s \in S$.

Theorem

There is a one-to-one correspondence between A-conjugacy classes of splittings of $A *_\theta G$ and elements of $H^1(G, A)$.

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THANK YOU!